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Exact BRS Symmetry realized on the Renormalization Group Flow

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Abstract

Using the average action defined with a continuum analog of the block spin transformation, we show the presence of gauge symmetry along the Wilsonian renormalization group flow. As a reflection of the gauge symmetry, the average action satisfies the quantum master equation(QME). We show that the quantum part of the master equation is naturally understood once the measure contribution under the BRS transformation is taken into account. Furthermore an effective BRS transformation acting on macroscopic fields may be defined from the QME. The average action is explicitly evaluated in terms of the saddle point approximation up to one-loop order. It is confirmed that the action satisfies the QME and the flow equation.

§1. Introduction

For the definition of Wilsonian effective action,[1)–3)] one needs to introduce some regularization. Therefore, it is a nontrivial problem if symmetries such as chiral or gauge symmetry can survive along the renormalization group (RG) flow, and if so how they can be realized in the effective theory.

An important contribution to see a (modified or broken) gauge symmetry on the RG flow is due to Ellwanger.[4)] He showed that there exists the broken Ward-Takahashi (WT) or Slavnov-Taylor identity along the flow expressed as $\Sigma_k = 0$ in his notation,[*)] where k denotes an IR cutoff. Once we find a theory on the hypersurface defined by $\Sigma_k = 0$ in the coupling space, it remains on the surface along the RG flow and in the limit of $k \to 0$ the identity reduces to the Zinn-Justin equation: the broken WT identity is, in this sense, connected to the usual WT identity. This viewpoint suggests that we could modify the gauge symmetry broken due to the regularization such that it could be connected smoothly to the usual gauge symmetry.

It had been long believed that the realization of a chiral symmetry was impossible on the lattice.[5)] However Lüscher[6)] took an important step by providing an exact chiral symmetry on the lattice decade after the Ginsparg-Wilson's paper.[7)] His chiral symmetry has a different form compared with the continuum chiral symmetry.

The above example may suggest the following possibility: a symmetry in a field theory survives even after a regularization, its form could be generally different from its familiar form. In our earlier publication,[8)] we pursued this possibility in the context of Wilsonian RG. We defined a procedure to give an effective field theory with an IR cutoff. In this setting it was shown that we may define a quantity similar to the Ellwanger's Σ_k: the equation $\Sigma_k = 0$ is found to be the quantum master equation (QME). We also constructed explicitly the symmetry transformation on the macroscopic fields, which was called as the renormalized transformation. With this result we claimed that a symmetry survives the regularization and is kept along the RG flow. We emphasize that the symmetry on the flow is exact and it is not "modified" or "broken". The Maxwell theory and the chiral symmetry were the two examples studied in Ref. 8). For the latter, we obtained continuum analogs of the Ginsparg-Wilson relation and the Lüscher's symmetry.

In the present paper we will show that our procedure may be naturally extended to an interacting gauge theory, typically the non-Abelian gauge theory coupled to any matter fields. A major difference from our earlier examples is the presence of the quantum part in the master equation. Although it had been regarded as a "breaking" term of the symmetry, we will

*) We use the same notation Σ_k for the corresponding quantity in our formulation.

see its presence is necessary to keep the symmetry. The renormalized BRS transformation is given as we did in our previous paper. To see more explicitly how our formulation works, we evaluate the average action with the saddle point approximation up to one-loop order: it will be shown that the action satisfies both the master equation and the flow equation.

This paper is organized as follows. In sect.2, after a brief explanation of Batalin-Vilkovisky (BV) antifield formalism,[9)*)] the average action is introduced and shown to satisfy the QME and the RG flow equation. For the BRS invariance of the average action, the quantum part of the master equation naturally emerges, which is the subject of sect.3. The renormalized BRS transformation is also given. In sect. 4 we evaluate the average action with the saddle point approximation. The last section is devoted to the summary and further discussions on the average action. Explanations of our notations will be found in the Appendix A. Some relations in sect. 4 are proved in the appendices B and C.

Owing to the presence of Grassmann odd fields, we have to keep track of signs carefully. In order to make equations correct and, at the same time, as simple as possible, we will introduce abbreviations whenever possible.

§2. The average action and its properties

The average action was introduced by Wetterich[11)] to realize a continuum analog of the block spin transformation. Before presenting it, let us describe the microscopic action and its properties in the antifield formalism.

2.1. *The antifield formalism*

In the following ϕ_a denotes all the fields in the system under consideration: eg, gauge, ghosts, antighosts, B-fields and matters for the non-Abelian theory. Further we introduce their antifields ϕ^*_a. For the gauge-fixing, we perform a canonical transformation: $\phi_a \to \phi_a, \quad \phi^*_a \to \phi^*_a + \partial\Psi/\partial\phi_a$, where Ψ is the gauge fermion, a function only of the fields. This gauge fixed basis is convenient, since it retains the antifields. Let $S_0[\phi]$ be a BRS invariant gauge fixed action in the new basis. We consider then an extended action, linear in the antifields:

$$S[\phi, \phi^*] \equiv S_0[\phi] + \phi^*\delta\phi. \tag{2·1}$$

Here $\delta\phi_a$ is the BRS transformation of ϕ_a. The full expression of the second term is given in eq.(A·2).

*) For reviews, see Ref. 10)

Under a set of BRS transformations,

$$\begin{aligned}\delta\phi_a &= \frac{\overrightarrow{\partial} S}{\partial\phi_a^*} = (-1)^{\epsilon_a+1}\frac{\overleftarrow{\partial} S}{\partial\phi_a^*},\\ \delta\phi_a^* &= -\frac{\overrightarrow{\partial} S}{\partial\phi_a} = (-1)^{\epsilon_a+1}\frac{\overleftarrow{\partial} S}{\partial\phi_a},\end{aligned} \tag{2·2}$$

the extended action $S[\phi,\phi^*]$ is shown to be invariant:

$$\delta S[\phi,\phi^*] = \delta S_0[\phi] + \phi^*\delta^2\phi + (-1)^{\epsilon_a+1}\delta\phi_a^*\delta\phi_a = 0, \tag{2·3}$$

where ϵ_a is the Grassmann parity of the field ϕ_a. The sign in the third term of eq.(2·3) appears since we have chosen the BRS transformation to act from the right. Another important sign appears in changing a right derivative to a left one and vice versa, as in (2·2). See (A·1) for a general formula.

With the antibracket,

$$(F,G)_\phi \equiv \frac{F\overleftarrow{\partial}}{\partial\phi}\frac{\overrightarrow{\partial} G}{\partial\phi^*} - \frac{F\overleftarrow{\partial}}{\partial\phi^*}\frac{\overrightarrow{\partial} G}{\partial\phi}, \tag{2·4}$$

the BRS transformation may be written as $\delta F \equiv (F,S)_\phi$. In terms of the antibracket gauge invariance of the action is nicely summarized as the master equation: $(S,S)_\phi = 0$. In eq.(2·4), the summation over indices and the momentum integration are implicit.

For the following discussion the action (2·1) is our starting point. So we assume that the action is linear in the antifield ϕ^*. This includes the Yang-Mills fields coupled to matter fields as a typical and important example. Actually our consideration may be extended to an action with nonlinear ϕ^* dependence, which will be discussed in Ref. 13).

2.2. *The average action*

The average action Γ_k, with an IR cutoff k, is written in terms of macroscopic fields Φ after integrating out the high frequency modes,

$$e^{-\Gamma_k[\Phi,\phi^*]/\hbar} = \int \mathcal{D}\phi e^{-S_k[\phi,\Phi,\phi^*]/\hbar}, \tag{2·5}$$

$$S_k[\phi,\Phi,\phi^*] = S_0[\phi] + \phi^*\delta\phi + \frac{1}{2}(\Phi - f_k\phi)\ R_k\ (\Phi - f_k\phi). \tag{2·6}$$

The third term on the rhs of eq.(2·6) is our abbreviated notation for the full expression given in eq.(A·3). The functions $f_k(p)$ and $R_k(p)$ should be chosen appropriately so that the macroscopic fields carry momentum less than k. Though we do not need their explicit forms in this paper, it would be instructive to see how the high frequency modes are integrated out in the above path integral.

To realize a continuum analog of the block spin transformation, Wetterich wrote down some criteria on the functions. For example,

$$f_k(p) = \exp\left(-\alpha\left(\frac{p^2}{k^2}\right)^\beta\right),$$
$$[R_k(p)]_{ab} = (1 - f_k^2(p))^{-1} \times [\mathcal{R}_k(p)]_{ab},$$

with positive α and β are the functions which satisfy the criteria (See Ref. 11) for details). The matrix $[\mathcal{R}_k(p)]_{ab}$ is at most polynomial in p.

Note that: 1) the function $f_k(p)$ is close to one for the momentum lower than k and decreases rapidly for the higher momentum; 2) consequently the factor $(1 - f_k^2(p))^{-1}$ in $R_k(p)$ is almost constant for high momentum and getting very large for the momentum lower than k, the p dependence of $[\mathcal{R}_k(p)]_{ab}$ adds only minor modulation to this behavior. This implies that $\Phi(p) \sim \phi(p)$ for $p < k$, while $\Phi(p)$ with $p > k$ does not carry any information of the microscopic dynamics and appears in a simple quadratic form in the average action. In the rest of the paper, we do not need the functions explicitly and only assume some properties: $[R_k(p)]_{ab} = (-)^{\epsilon_a\epsilon_b}[R_k(-p)]_{ba}$; the components of R_k vanish for mixed Grassmann parity indices.

2.3. *The quantum master equation*

An important question is: how the gauge symmetry at the microscopic level reflects in $\Gamma_k[\Phi, \phi^*]$? The answer was given in our earlier paper:[8] the macroscopic action satisfies the QME.

The BRS invariance of the microscopic action may be written as,

$$\int \mathcal{D}\phi e^{-S_k[\phi+\delta\phi\lambda,\Phi,\phi^*]/\hbar} - \int \mathcal{D}\phi e^{-S_k[\phi,\Phi,\phi^*]/\hbar} = 0, \tag{2·7}$$

with the Grassmann odd parameter λ. We assumed the BRS invariance of the measure, $\mathcal{D}\phi$; thus anomalies are not considered here. Rewriting eq.(2·7), we obtain

$$0 = \hbar^2 e^{\Gamma[\Phi,\phi^*]_k/\hbar}\Delta_\Phi e^{-\Gamma[\Phi,\phi^*]_k/\hbar} \equiv \Sigma[\Phi, \phi^*]_k,$$
$$\Delta_\Phi \equiv \sum_a (-)^{\epsilon_a+1}\int \mathrm{d}p f_k(p)\frac{\partial^r}{\partial\Phi_a(-p)}\frac{\partial^r}{\partial\phi_a^*(p)},$$

or

$$\Sigma_k[\Phi, \phi^*] = \frac{1}{2}(\Gamma_k[\Phi, \phi^*], \Gamma_k[\Phi, \phi^*])_\Phi - \hbar\Delta_\Phi\Gamma_k[\Phi, \phi^*] = 0. \tag{2·8}$$

Here the bracket is defined in terms of Φ and ϕ^*:

$$(F, G)_\Phi \equiv \int \mathrm{d}^4p f_k(p)[\frac{F\overleftarrow{\partial}}{\partial\Phi_a(-p)}\frac{\overrightarrow{\partial}G}{\partial\phi_a^*(p)} - \frac{F\overleftarrow{\partial}}{\partial\phi_a^*(-p)}\frac{\overrightarrow{\partial}G}{\partial\Phi_a(p)}]. \tag{2·9}$$

The comparison of eqs.(2·4) and (2·9) suggests that ϕ^*/f_k may be regarded as the antifield associated with Φ.

2.4. *The flow equation for the average action*

A straightforward calculation leads us to the flow equation:

$$\hbar\partial_k e^{-\Gamma_k[\Phi,\phi^*]/\hbar} = -\left[X + \frac{\hbar}{2}\mathrm{Str}(R_k^{-1}\partial_k R_k) + \hbar\ \mathrm{Str}(\partial_k(\ln f_k))\right] e^{-\Gamma_k[\Phi,\phi^*]/\hbar}, \tag{2·10}$$

$$X \equiv -\frac{\hbar^2}{2}\frac{\partial^l}{\partial\Phi}(\partial_k R_k^{-1})\frac{\partial^r}{\partial\Phi} + \partial_k(\ln f_k)\left[\hbar^2\frac{\partial^l}{\partial\Phi}R_k^{-1}\frac{\partial^r}{\partial\Phi} + \hbar\Phi\frac{\partial^l}{\partial\Phi}\right]. \tag{2·11}$$

Here we used the fact, $(R_k)_{\text{even odd}} = (R_k)_{\text{odd even}} = 0$, in our choice for R_k.

An interesting property of the quantity $\Sigma_k[\Phi,\phi^*]$ was found by Ellwanger:[4] using the flow equation (2·10) we may show the following,

$$\hbar\partial_k\Sigma_k = (e^{\Gamma_k/\hbar}Xe^{-\Gamma_k/\hbar})\Sigma_k - e^{\Gamma_k/\hbar}X\ (e^{-\Gamma_k/\hbar}\Sigma_k). \tag{2·12}$$

Therefore once we are on the hypersurface $\Sigma_k = 0$ in the coupling space, we will keep the same condition even if we change the IR cutoff k.

§3. The QME and the renormalized BRS transformation

In earlier works it had been generally understood that the momentum cutoff breaks gauge invariance; we only have the condition so that the gauge invariance recovers when the IR cutoff is removed. The condition was beautifully summarized in Ref. 4) and its connection to the QME was clarified in our earlier paper.[8] The commonly shared view is that terms corresponding to $\Delta_\Phi\Gamma_k$ represent the breaking of the gauge invariance.[*] Here we show that the BRS invariance will be kept *including* $\Delta_\Phi\Gamma_k$ term.

In the following we first explain how a QME is related to the BRS invariance of a generic gauge invariant system. One finds the variation of the path integral measure is exactly the $\Delta_\Phi\Gamma_k$ term. Based on this understanding we may define the renormalized BRS transformation for the macroscopic fields.

3.1. *A generic gauge system*

Let us consider a generic gauge system with the action $\mathcal{A}[\eta,\eta^*]$, where (η,η^*) could be the microscopic fields (ϕ,ϕ^*) or the macroscopic fields (Φ,ϕ^*). Under the transformation,

$$\eta' = \eta + \delta\eta\lambda,$$

[*] If one uses the average action, the condition is written in a very simple form as QME. Of course, in other formalisms it looks completely different and the "breaking terms" look very different in their appearances.

$$\delta\eta = (\eta, \mathcal{A})_\eta = \frac{\overrightarrow{\partial}\mathcal{A}}{\partial\eta^*},$$

we require that the path integral be invariant,

$$\int \mathcal{D}\eta e^{-\mathcal{A}[\eta,\eta^*]/\hbar} = \int \mathcal{D}\eta' e^{-\mathcal{A}[\eta',\eta^*]/\hbar},$$

where λ is the transformation parameter. The BRS invariance of the path integral including the measure may be written as $\delta(\mathcal{A}[\eta,\eta^*] - \hbar \ln \mathcal{D}\eta) = 0$. We will presently see that this is nothing but a QME and its quantum part is due to the variation of the measure.

Let us look at the first term in the above mentioned equation,

$$\delta\mathcal{A}[\eta,\eta^*] = \frac{\mathcal{A}\overleftarrow{\partial}}{\partial\eta}\delta\eta = \frac{\mathcal{A}\overleftarrow{\partial}}{\partial\eta}\frac{\overrightarrow{\partial}\mathcal{A}}{\partial\eta^*} = \frac{1}{2}(\mathcal{A},\mathcal{A})_\eta.$$

If we assume that the path integral measure is flat, $\mathcal{D}\eta = \prod_a \mathrm{d}\eta_a$, the logarithm of the measure transforms as $\ln \mathcal{D}\eta' = \ln \mathcal{D}\eta + (\delta \ln \mathcal{D}\eta)\lambda$,*)

$$(\delta \ln \mathcal{D}\eta)\lambda = \ln\ \mathrm{Sdet}\frac{\partial^r}{\partial\eta_a}(\eta + \frac{\partial^l\mathcal{A}}{\partial\eta^*}\lambda)_b \sim \frac{\overrightarrow{\partial}}{\partial\eta^*_a}\mathcal{A}[\eta,\eta^*]\frac{\overleftarrow{\partial}}{\partial\eta_a}\lambda. \tag{3·1}$$

Therefore including the contribution from the measure, we obtain the QME,

$$\frac{1}{2}(\mathcal{A}[\eta,\eta^*],\mathcal{A}[\eta,\eta^*])_\eta - \hbar\frac{\overrightarrow{\partial}}{\partial\eta^*_a}\mathcal{A}[\eta,\eta^*]\frac{\overleftarrow{\partial}}{\partial\eta_a} = 0. \tag{3·2}$$

3.2. *The average action*

Consider the following path integral,

$$\int \mathcal{D}\phi e^{-S[\phi,\phi^*]/\hbar} \tag{3·3}$$

$$= \int \mathcal{D}\Phi\mathcal{D}\phi\ e^{-S_k[\phi,\Phi,\phi^*]/\hbar} = \int \mathcal{D}\Phi e^{-\Gamma_k[\Phi,\phi^*]/\hbar}, \tag{3·4}$$

To the original path integral we insert the gaussian integration with respect to Φ and reverse the order of the integrations, then we find the path integral over the average action with the flat measure for Φ-integration. The gauge symmetry of the original system is expressed as the classical master equation. The path integral of the average action carries the same information. As evident from our general argument, the symmetry is expressed as the QME with its quantum part $\Delta_\Phi\Gamma_k$ coming from the transformation of the path integral measure.

*) The argument of eq.(3·1) is adapted from Ref. 12).

3.3. *The renormalized BRS transformation*

From the above argument we see that the renormalized BRS transformation may be read off from the classical part of the QME :[8]

$$\delta_r \Phi \equiv f_k \frac{\overrightarrow{\partial} \Gamma_k}{\partial \phi^*} = f_k \langle \delta\phi \rangle_\phi, \tag{3·5}$$

$$\delta_r \phi^* \equiv -f_k \frac{\overrightarrow{\partial} \Gamma_k}{\partial \Phi} = -\langle f_k R_k (\Phi - f_k \phi) \rangle_\phi. \tag{3·6}$$

Here we used the notation,

$$\langle \mathcal{O} \rangle_\phi = \int \mathcal{D}\phi \; \mathcal{O} \; e^{-S_k/\hbar} \Big/ \int \mathcal{D}\phi \; e^{-S_k/\hbar}. \tag{3·7}$$

In Ref. 14) the cutoff dependent BRS transformation was considered in a different approach.

Some comments are in order. Firstly, let us emphasize that the quantum part had long been understood to suggest the breaking of the gauge symmetry, which is not the correct understanding from our viewpoint. Secondly, as far as we know of, this is the second example where the quantum part of a QME plays an important role; the first one was the string field theory(SFT).[12] It is probably very important to remember the QME is deeply related to the unitarity of the SFT.

§4. The average action in the saddle point approximation

It would be usually impossible to fully evaluate the path integral (2·5) to construct an average action. In order to understand the formalism in more concrete terms, a systematic evaluation of the average action in (2·5) is definitely instructive. The loop expansion with the saddle point method suits for our purpose: it provides a way to integrate out high frequency modes systematically. In this section we will calculate the average action up to one-loop order.

The saddle point, $\phi(p) = \phi_0(p)$, is determined by the following equation,

$$-f_k R_k (\Phi - f_k \phi_0) + \frac{\overrightarrow{\partial} (\phi_a^* P_a[\phi_0] + S_0[\phi_0])}{\partial \phi_0} = 0, \tag{4·1}$$

where $P_a[\phi]$ denotes the BRS transformation of ϕ_a: $P_a[\phi] \equiv \delta\phi_a$. The saddle point equation gives an implicit function, $\phi_0 = \phi_0[\Phi, \phi^*]$. Note that in eq.(4·1) we have omitted the indices and the momentum dependence for simplicity. The left derivative, $\overrightarrow{\partial}/\partial\phi_0$, in the second term is taken with ϕ^* fixed.

Now the average action at the tree level is given as,

$$\Gamma_k^{(0)}[\Phi, \phi^*] \equiv S_k[\phi_0[\Phi, \phi^*], \Phi, \phi^*], \tag{4·2}$$

and the one-loop correction is the superdeterminant,

$$\Gamma_k^{(1)}[\Phi, \phi^*] = \frac{\hbar}{2} \ln \mathrm{Sdet}(A[\phi_0, \phi^*]), \tag{4·3}$$

of the matrix A,

$$A_{ab}[\phi_0, \phi^*] = f_k^2 [R_k]_{ab} + \frac{\overrightarrow{\partial}}{\partial \phi_0^a} (\phi_c^* P_c[\phi_0] + S_0[\phi_0]) \frac{\overleftarrow{\partial}}{\partial \phi_0^b}. \tag{4·4}$$

Let us see the one-loop average action, $\Gamma_k^{(0)} + \Gamma_k^{(1)}$, satisfies both the QME and the flow equation.

4.1. *The one-loop QME*

The QME to be proved may be rewritten as:

$$(\Gamma_k^{(0)}, \Gamma_k^{(0)})_\Phi = 0, \tag{4·5}$$
$$(\Gamma_k^{(0)}, \Gamma_k^{(1)})_\Phi - \hbar \Delta_\Phi \Gamma_k^{(0)} = 0, \tag{4·6}$$

where we have used the fact, $\frac{1}{2}(\Gamma_k^{(0)}, \Gamma_k^{(1)})_\Phi = \frac{1}{2}(\Gamma_k^{(1)}, \Gamma_k^{(0)})_\Phi$, which is easily seen by using $\overrightarrow{\partial} \Gamma_k^{(0)}/\partial \Phi_a = (-1)^{\epsilon_a} \Gamma_k^{(0)} \overleftarrow{\partial}/\partial \Phi_a$ etc.

The tree level master equation (4·5) may be confirmed by using the tree level renormalized BRS transformations for Φ and ϕ^*:

$$\delta_r^{(0)} \Phi = (\Phi, \Gamma_k^{(0)})_\Phi = f_k P[\phi_0] \tag{4·7}$$

$$\begin{aligned} \delta_r^{(0)} \phi^* &= (\phi^*, \Gamma_k^{(0)})_\Phi = -f_k R_k (\Phi - f_k \phi_0) \\ &= -\frac{\overrightarrow{\partial} (\phi_a^* P_a[\phi_0] + S_0[\phi_0])}{\partial \phi_0}. \end{aligned} \tag{4·8}$$

The final expression in eq.(4·8) follows from the saddle point equation (4·1). Further, using eqs.(4·7) and (4·8), we may obtain the transformation of the implicit function $\phi_0^a[\Phi, \phi^*]$:

$$\begin{aligned} \delta_r^{(0)} \phi_0^a[\Phi, \phi^*] &= (\phi_0 \overleftarrow{\partial}/\partial \Phi)_{ab} \delta_r^{(0)} \Phi_b + (\phi_0 \overleftarrow{\partial}/\partial \phi^*)_{ab} \delta_r^{(0)} \phi_b^* \\ &= P_a[\phi_0], \end{aligned} \tag{4·9}$$

which will be shown in the Appendix B. From (4·2) we see that $\Gamma_k^{(0)}$ may be written as the rhs of (2·6) with ϕ replaced by ϕ_0. It is easy to see that the first and second terms of that expression are invariant under eqs.(4·8) and (4·9); the third term of it is also invariant under

eqs.(4·7) and (4·9). Therefore $\mathit{\Gamma}_k^{(0)}$ is invariant under (4·7), (4·8) and (4·9): this proves the tree level master equation (4·5).

In the Appendix C we will show that the lhs of (4·6) reduces to

$$\begin{aligned}&(\mathit{\Gamma}_k^{(0)}, \mathit{\Gamma}_k^{(1)})_{\mathit{\Phi}} - \hbar \mathit{\Delta}_{\mathit{\Phi}} \mathit{\Gamma}_k^{(0)} \\ &= \frac{\hbar}{2}(A^{-1})_{ab} \frac{\overrightarrow{\partial}}{\partial \phi_0^b}[(\phi_d^* P_d + S_0)\frac{\overleftarrow{\partial}}{\partial \phi_0^c} P_c]\frac{\overleftarrow{\partial}}{\partial \phi_0^a} - \hbar \frac{\overrightarrow{\partial}}{\partial \phi_a^*} S[\phi_0, \phi^*]\frac{\overleftarrow{\partial}}{\partial \phi_0^a},\end{aligned} \tag{4·10}$$

where $S[\phi_0, \phi^*]$ is the extended action (2·1) evaluated at the saddle point. The first term of eq.(4·10) vanishes owing to the relations,

$$\frac{S_0[\phi_0]\overleftarrow{\partial}}{\partial \phi_0^a} P_a[\phi_0] = 0, \tag{4·11}$$

$$\frac{P_a[\phi_0]\overleftarrow{\partial}}{\partial \phi_0^b} P_b[\phi_0] = 0. \tag{4·12}$$

These respectively come from the BRS invariance of the action S_0 and the nilpotency of the BRS transformation at the microscopic level. Similarly it is easy to observe that the second term of eq.(4·10) is nothing but the quantum part of the QME for $S[\phi, \phi^*]$; it vanishes since we assumed that the measure $\mathcal{D}\phi$ is BRS invariant.

4.2. *The flow equation for the one-loop average action*

Let us see that the one-loop average action satisfies the flow equation as well. This is a consistency check of our calculation.

$$\begin{aligned}&-\partial_k \mathit{\Gamma}_k + e^{\mathit{\Gamma}_k/\hbar}[X + \frac{\hbar}{2}\mathrm{Str}(R_k^{-1}\partial_k R_k) + \hbar \mathrm{Str}(\partial_k(\ln f_k))]e^{-\mathit{\Gamma}_k/\hbar} \\ &\sim -\frac{\mathit{\Gamma}_k^{(1)}\overleftarrow{\partial}}{\partial \phi_0}\partial_k \phi_0 - \frac{\mathit{\Gamma}_k^{(1)}\overleftarrow{\partial}}{\partial \mathit{\Phi}}[(\partial_k R_k^{-1})R_k(\mathit{\Phi} - f_k \phi_0) - \partial_k(\ln f_k)(\mathit{\Phi} - 2 f_k \phi_0)].\end{aligned} \tag{4·13}$$

The cancellation of $\mathrm{O}(\hbar^0)$ terms follows trivially; thus here on the rhs we wrote only $\mathrm{O}(\hbar)$ terms. Remember that $\mathit{\Gamma}_k^{(1)}$ depends on $\mathit{\Phi}$ only through its ϕ_0 dependence. So one may rewrite the $\mathit{\Phi}$ derivative of (4·13) into ϕ_0 derivative; then using (B·1) and the relation,

$$-\partial_k f_k R_k(\mathit{\Phi} - 2 f_k \phi_0) - f_k \partial_k R_k(\mathit{\Phi} - f_k \phi_0) + A \partial_k \phi_0 = 0, \tag{4·14}$$

the vanishing of the rhs of (4·13) follows. The relation (4·14) is obtained by differentiating the saddle point equation.

§5. Summary and Discussions

By using the average action formalism, we have shown that our claim in our earlier publication [8)] may be justified even for an interacting gauge theory: ie, a gauge symmetry survives

even with the presence of a cutoff and the corresponding renormalized BRS transformation may be constructed from the QME.

The average action satisfies the QME if the original classical action is gauge invariant. At this point we have noticed that the antifield formalism is very convenient to describe the symmetry property of the average action. It also follows the flow equation, which also implies that once the system satisfies the WT identity with some IR cutoff it will remain so along the RG flow.

The saddle point evaluation is performed for the average action up to the one-loop order. The QME and the flow equation are confirmed explicitly. As we have seen above, there is no essential difficulty to extend our analysis to higher orders. It would be worth pointing out that the construction of an action satisfying both equations had not been done earlier. A related calculation is due to Ellwanger:[4] the gauge mass term was obtained from the master and flow equations independently and found to be coincide.

The quantum part of a QME had been regarded as an obstacle for the gauge symmetry. We have shown that it is necessary for the symmetry since the measure is not invariant under the renormalized BRS transformation: the jacobian under the transformation is exactly the quantum part of the QME. This argument implies also that we may read off the renormalized BRS transformation as we did earlier for free field theories. The transformation for the averaged field is particularly simple: $\delta_r \Phi = f_k \langle \delta\phi \rangle_\phi$. Similarly the quantity Σ_k defined referring to the cutoff scale k is also expressed as a path integral average. Let us explain briefly how it is so in the following.

To be observed shortly our argument is applicable even for a microscopic action with symmetry breaking terms or anomalies. So let us consider for the moment the average action $\Gamma_k[\Phi,\ \Phi^*]$ defined with eq.(2·5), but with an action $S[\phi,\ \phi^*]$ which is not necessarily BRS invariant. For the microscopic fields, we define the quantity Σ as,

$$\Sigma[\phi,\ \phi^*] \equiv \frac{1}{2}\left(S,\ S\right)_\phi - \hbar \Delta_\phi S = \hbar^2 \exp(S/\hbar) \Delta_\phi \exp(-S/\hbar).$$

The functional average of it may be rewritten as

$$\begin{aligned} \langle \Sigma[\phi,\ \phi^*] \rangle_\phi &= \hbar^2 e^{\Gamma_k/\hbar} \int \mathcal{D}\phi\ e^{(S-S_k)/\hbar} \left(\Delta_\phi e^{-S/\hbar} \right) \\ &= \hbar^2 e^{\Gamma_k/\hbar} \Delta_\Phi e^{-\Gamma_k/\hbar} \equiv \Sigma_k[\Phi, \phi^*]. \end{aligned} \tag{5·1}$$

For $S[\phi,\ \phi^*]$ which does satisfy the (classical) master equation, eq.(5·1) tells us the average action satisfy the QME, $\Sigma_k[\Phi, \phi^*] = 0$. This is an important result: the QME for the average action is obtained from the master equation for the microscopic action. Note that the relation (5·1) holds even for the case that Σ does not vanish, which must have further

implications. For example, it tells us how a symmetry breaking term changes along the RG flow.

In our formulation, there remain a couple of questions to be clarified. Among others the following two are particularly important: 1) whether our QME reduces to the usual Zinn-Justin equation in the limit of $k \to 0$; 2) how we prepare the UV theory. In the forthcoming paper [13] we will show that the approach presented here may be extended to most general gauge theories. The relations to other approaches [14) - 17)] will be given as well; at the same time it will be explained how the Zinn-Justin equation is realized in the limit of $k \to 0$. The second question will be discussed by introducing an UV cutoff Λ and imposing appropriate boundary conditions on the average action.

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Appendix A

—— *On notations* ——

The left and right derivatives are written as:

$$\frac{\overrightarrow{\partial} F}{\partial \phi} \equiv \frac{\partial^l F}{\partial \phi}, \quad \frac{F \overleftarrow{\partial}}{\partial \phi} \equiv \frac{\partial^r F}{\partial \phi}.$$

We find that the notations on the lhs provide us with simpler expressions for many equations. However whenever convenient to avoid possible confusion, we use those on the rhs.

The sign associated with the change from a right derivative to a left derivative or vice versa is very important,

$$\frac{F \overleftarrow{\partial}}{\partial \chi} = (-1)^{\epsilon_\chi(\epsilon_F + 1)} \frac{\overrightarrow{\partial} F}{\partial \chi}. \tag{A·1}$$

Here we explain our abbreviated notations for some examples. The second term of $S[\phi, \phi^*] \equiv S_0[\phi] + \phi^* \delta\phi$ is the shorthand notation for

$$\phi^* \delta \phi \equiv \sum_a \int \mathrm{d}^4 p \phi^*_a(-p) \delta\phi_a(p). \tag{A·2}$$

In the multiplication on the lhs the summation over the index a and the momentum integration are implicit. Similarly in the block spin transformation we use the following,

$$(\Phi - f_k\phi)R_k(\Phi - f_k\phi) \equiv \int \mathrm{d}^4p(\Phi - f_k\phi)^a(-p)[R_k(p)]_{ab}(\Phi - f_k\phi)^b(p). \qquad (\mathrm{A}\cdot 3)$$

Appendix B

—— A proof of eq.(4·9): $\delta_r^{(0)}\phi_0 = P[\phi_0]$ ——

Here we show the following equation:

$$P[\phi_0] = (\partial^r\phi_0/\partial\Phi)\delta_r^{(0)}\Phi + (\partial^r\phi_0/\partial\phi^*)\delta_r^{(0)}\phi^*.$$

By differentiating the saddle point equation (4·1) with respect to Φ and ϕ^*, we obtain the relations,

$$f_kR_k = A(\partial^r\phi_0/\partial\Phi), \qquad (\mathrm{B}\cdot 1)$$

$$\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_a^*P_a)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0} + A(\partial^r\phi_0/\partial\phi^*) = 0, \qquad (\mathrm{B}\cdot 2)$$

where $A_{ab}[\phi_0,\phi^*]$ is defined in eq.(4·4). In eq.(B·2), the ϕ^* derivative in the first term is taken with ϕ_0 fixed, which is denoted by the subscript ϕ_0.

Using them and the tree level renormalized BRS transformation, the equation to be proved may be rewritten as,

$$P = A^{-1}f_k^2R_kP + A^{-1}\left[\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_a^*P_a)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0}\right]\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_b^*P_b + S_0).$$

Let us see the vanishing of the difference of lhs and rhs multiplied by A,

$$\begin{aligned}
&(A - f_k^2R_k)_{ab}P_b - \left[\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^*P_c)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0}\right]_{ab}\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^*P_d + S_0)\\
&= \left[\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^*P_c + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0}\right]_{ab}P_b - \left[\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^*P_c)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0}\right]_{ab}\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^*P_d + S_0),
\end{aligned} \qquad (\mathrm{B}\cdot 3)$$

where we substituted eq.(4·4) on the rhs. Taking the ϕ^*-differentiation, we rewrite the second term;

$$\begin{aligned}
&-\left[\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^*P_c)\frac{\overleftarrow{\partial}}{\partial\phi^*}\bigg|_{\phi_0}\right]_{ab}\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^*P_d + S_0)\\
&= -\left(\frac{\overrightarrow{\partial}}{\partial\phi_0^a}P_b(-)^{\epsilon_b+1}\right)\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^*P_d + S_0)\\
&= \left(\frac{\overrightarrow{\partial}P_b}{\partial\phi_0^a}\right)\left[(\phi_d^*P_d + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0^b}\right] = (-)^{\epsilon_a\epsilon_b}\left[(\phi_d^*P_d + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0^b}\right]\left(\frac{\overrightarrow{\partial}P_b}{\partial\phi_0^a}\right).
\end{aligned}$$

Thus the rhs of (B·3) may be rewritten as,

$$\frac{\overrightarrow{\partial}}{\partial\phi_0^a}\left([\phi_c^* P_c + S_0]\frac{\overleftarrow{\partial}}{\partial\phi_0^b}P_b\right),$$

which vanishes owing to eqs.(4·11) and (4·12).

Appendix C

—— A proof of eq.(4·10): the QME to one-loop order ——

In (4·10) the first term is the variation of $\Gamma_k^{(1)}$ by the tree level BRS transformation given in (4·7) and (4·8):

$$\begin{aligned}&(\ \Gamma_k^{(0)}, \Gamma_k^{(1)})_\Phi - \hbar\Delta_\Phi\Gamma_k^{(0)}\\ &= \frac{\hbar}{2}\mathrm{Str}A^{-1}\left(\left(\left.\frac{A\overleftarrow{\partial}}{\partial\phi_a^*}\right|_{\phi_0}\right)\delta_r^{(0)}\phi_a^* + \left(\frac{A\overleftarrow{\partial}}{\partial\phi_0^a}\right)\delta_r^{(0)}\phi_0^a\right) - \hbar\mathrm{tr}(f_k\partial^r P/\partial\Phi).\end{aligned} \qquad \text{(C·1)}$$

Since the matrix A is a function of ϕ_0 and ϕ^*, the variation under the tree level BRS transformation is taken with respect to those variables. The derivatives in the first term of (C·1) should be understood accordingly. The second term is the trace (not the supertrace) of the matrix $\partial^r P_a/\partial\Phi_b$, which may be rewritten by using eqs. (4·4) and (B·1) as

$$\begin{aligned}\Delta_\Phi\Gamma_k^{(0)} &= \mathrm{tr}(f_k\partial^r P/\partial\phi_0\cdot\partial^r\phi_0/\partial\Phi))\\ &= \mathrm{tr}(\partial^r P/\partial\phi_0 A^{-1}f_k^2 R_k)\\ &= \mathrm{tr}\left(\partial^r P/\partial\phi_0 A^{-1}[A - \frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^* P_c + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0}]\right).\end{aligned}$$

Therefore eq.(C·1) becomes,

$$\begin{aligned}&(\Gamma_k^{(0)}, \Gamma_k^{(1)})_\Phi - \hbar\Delta_\Phi\Gamma_k^{(0)}\\ &\quad = -\hbar\mathrm{tr}(\partial^r P/\partial\phi_0) + \frac{\hbar}{2}\mathrm{Str}A^{-1}\left(-\left(\left.\frac{A\overleftarrow{\partial}}{\partial\phi_a^*}\right|_{\phi_0}\right)\frac{\overrightarrow{\partial}}{\partial\phi_0^a}(\phi_c^* P_c + S_0) + \left(\frac{A\overleftarrow{\partial}}{\partial\phi_0^a}\right)P_a\right)\\ &\quad + \hbar\ \mathrm{tr}\left[A^{-1}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0}(\phi_c^* P_c + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0}\right)\partial^r P/\partial\phi_0\right].\end{aligned}$$

We may write the rhs more explicitly. After the ϕ^*-differentiation, the second and third terms are written as

$$\begin{aligned}&\frac{\hbar}{2}(-)^{\epsilon_a}A_{ab}^{-1}\left(-(-)^{(\epsilon_c+1)(\epsilon_a+1)}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0^b}P_c\frac{\overleftarrow{\partial}}{\partial\phi_0^a}\right)\frac{\overrightarrow{\partial}}{\partial\phi_0^c}(\phi_d^* P_d + S_0) + \left(\frac{A_{ba}\overleftarrow{\partial}}{\partial\phi_0^c}\right)P_c\right)\\ &+ \hbar A_{ab}^{-1}\left(\frac{\overrightarrow{\partial}}{\partial\phi_0^b}(\phi_d^* P_d + S_0)\frac{\overleftarrow{\partial}}{\partial\phi_0^c}\right)\left(\frac{P_c\overleftarrow{\partial}}{\partial\phi_0^a}\right).\end{aligned}$$

An easy calculation leads us to eq.(4·10): one must take care of signs carefully, in particular, those coming from eq.(A·1).

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